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Calculations of generalized Kervaire's invariant

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Abstract

Calculations of Browder's generalized Kervaire's invariant have been performed. By relating Browder's and Brown's invariants a preferred orientation has been brought to bear on the calculations. A product formula has been derived and is simplified using preferred orientations.

Keywords: Generalized Kervaire's invariant; Wu-orientation; Spectral orientation; Functional Steenrod squares; Stable homotopy

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0. Introduction

Kervaire's invariant is generalized by Browder [2] and Brown [4] (see Section 1 for a quick recollection of these generalizations). While Brown's invariant is always defined Browder's invariant is not always defined. Brown has given a product formula for his generalization of the Kervaire invariant in [5]. A product formula for the Browder's generalization of the Kervaire invariant in its most general form has eluded mathematicians until today. To make the point clear, let (M_1, c_1) and (M_2, c_2) be two Wu-oriented manifolds, let e be the product Wu-orientation (see [3]). Then a general formula relating Browder's generalized Kervaire invariants $k(M_1, c_1)$, $k(M_2, c_2)$ and $k(M_1 \times M_2, e)$ is not known until today. The difficulty

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lies in the decomposition of the functional Steenrod square used in the definition of $k(M_1 \times M_2, e)$ into the functional Steenrod squares used in the definitions of $k(M_i, c_i)$, $i = 1, 2$.

One way to avoid this difficulty is to define an invariant $k(M_1 \times M_2, c_1 \times c_2)$ in the same way as Browder defines $k(M_1 \times M_2, e)$, and to give a formula relating this to $k(M_i, c_i)$, $i = 1, 2$ (see Theorems 2.5 and 2.6). This approach can be justified by producing examples of manifolds for which $k(M_1 \times M_2, e) = (M_1 \times M_2, c_1 \times c_2)$ (see Corollaries 2.7–2.9).

Another approach to avoid the difficulty would be to try to choose suitable Wu-orientations, for example the preferred Wu-orientation as used by Brown in his approach to generalized Kervaire's invariant. To use the preferred orientation in Browder's setting one needs a link between Browder's and Brown's generalized Kervaire's invariant, which is known for framed manifolds [6]. So we first establish this link in the general setting and then proceed with our calculations. As a dividend of using preferred orientations we get a neat product formula.

The paper is arranged as follows. In Section 1 we give a quick recollection of Browder's and Brown's generalizations of the Kervaire invariant, change of orientations and preferred orientations. In Section 2 we state the main results. In Section 3 we prove Theorems 2.4 and 2.10–2.14. In Section 4 we prove Proposition 2.1, Theorem 2.2, and Corollary 2.3. We indicate the proofs of Theorems 2.5 and 2.6, and Corollaries 2.7–2.9 in Section 5.

1. A quick recollection of Browder's and Brown's generalizations of the Kervaire invariant, change of Wu-orientations and the preferred orientation

To cut short the presentation we refer the reader to [1–5] for the necessary details of the notations we are going to use in the following discussion.

Let M^{2n} be a smooth manifold (Poincaré duality space). As the Wu-classes, $v_i(M)$, are zero $\forall i \geqslant n+1$, the normal bundle (Spivak normal fibre space) of M, $\nu(M)$, has orientations in Wu$\langle v_{n+1}\rangle$-theory and also in Wu$\langle n\rangle$-theory, where the Wu$\langle v_{n+1}\rangle$-theory of bundles is constructed by killing the Wu-classes v_{n+1} of the universal bundle, and the Wu$\langle n\rangle$-theory of bundles is constructed by killing Wu-classes v_i, $\forall i \geqslant n+1$, of the universal bundle.

If (M^{2n}, c) is a manifold with an orientation c of its normal bundle in Wu$\langle v_{n+1}\rangle$-theory, one can define a spectral orientation map

$$g : W/\partial W \to \Sigma^k M_+,$$

where $(W, \partial W)$ is a manifold with boundary of dimension $2n+k$ obtained by a universal construction (see [2, p. 195]); for $\mathrm{Sq}^{n+1}(x) \in H^{2n+k}(W, \partial W; \mathbb{Z}/2)$ it satisfies $\mathrm{Sq}^{n+1}(x) = 0$, for every x. Let the map

$$\Psi : \left[\mathrm{Ker}\left(g^* \circ \Sigma^k\right)\right]^n \to \mathbb{Z}/2 \tag{1.1}$$

be defined by $\Psi(x) = [\mathrm{Sq}^{n+1}_{((\Sigma^k\phi)\circ g)}(\Sigma^k\iota)][W]$, where $[W]$ is the fundamental class of $(W, \partial W)$, and the functional Steenrod square $\mathrm{Sq}^{n+1}_{((\Sigma^k\phi)\circ g)}$ is defined with zero indeterminacy for the composite map

$$W/\partial W \to \Sigma^k M_+ \to \Sigma^k K(\mathbb{Z}/2, n)_+,$$

where ϕ represents x and ι is the fundamental class of $K(\mathbb{Z}/2, n)$. Ψ is a $\mathbb{Z}/2$-quadratic form and

$$(x, y) = \Psi(x+y) - \Psi(x) - \Psi(y)$$

is the bilinear form obtained from the cup product pairing. Henceforth Ψ *will be referred to as Browder's quadratic form*. The generalized Kervaire invariant of Browder, $k(M, c)$, will be defined as

$$k(M, c) = \mathrm{Arf}(\Psi), \quad \text{if } (x, y) \text{ is nonsingular.} \tag{1.2}$$

Condition 1.1. If (x, y) is not nonsingular, then $k(M, c)$ is defined if and only if $\Psi \mid R \equiv 0$, where R is the radical of the bilinear form (x, y) restricted to the domain of Ψ.

If Condition 1.1 is satisfied, then $k(M, c) = \mathrm{Arf}(\overline{\Psi})$, where

$$\overline{\Psi} : \left[\mathrm{Ker}(g^* \circ \Sigma^k)\right]^n / R \to \mathbb{Z}/2$$

is the quotient of Ψ. R is calculated by Browder [2, (2.2)]; it is given by

$$R = \left[\mathrm{Ker}(g^* \circ \Sigma^k)\right]^n \cap c_W^* H^n(W; \mathbb{Z}/2) \tag{1.3}$$

where $c_W : M \to W$ is induced by c.

Brown, on the other hand, shows that there exist homomorphisms

$$h : \left\{S^{2n+k}, T(\gamma\langle n\rangle) \wedge K(\mathbb{Z}/2, n)\right\} \to \mathbb{Z}/4$$

satisfying $h(\lambda) = 2$, where λ is the image of the generator of $\{S^{2n+k}, S^k \wedge K_n\}$ under the map induced by the inclusion of the fibre of $\gamma\langle n\rangle$. The symbol $\{\cdot, \cdot\}$ stands for the set of stable homotopy classes of maps, γ for the universal bundle (universal spherical fibre space), and $T(\ )$ for the Thom space.

For each such h Brown defines a map

$$\Psi_h : H^n(M; \mathbb{Z}/2) \to \mathbb{Z}/4 \tag{1.4}$$

given by the composite

$$\begin{aligned} H^n(M; \mathbb{Z}/2) \to \left\{M_+, K(\mathbb{Z}/2, n)\right\} &\xrightarrow{A_\alpha} \left\{S^{2n+k}, T(\nu(M) \wedge K(\mathbb{Z}/2, n)\right\} \\ &\xrightarrow{T(c)_*} \left\{S^{2n+k}, T(\gamma\langle n\rangle) \wedge K(\mathbb{Z}/2, n)\right\} \xrightarrow{h} \mathbb{Z}/4, \end{aligned}$$

where $\alpha \in \pi_{2n+k}(T(\nu(M))$ is a spherical class of $T(\nu(M))$,

$$A_\alpha : \left\{M_+, K(\mathbb{Z}/2, n)\right\} \cong \left\{S^{2n+k}, T(\nu(M) \wedge K(\mathbb{Z}/2, n)\right\}$$

is defined by

$$A_\alpha\{f\} = \{(\mathrm{id} \wedge f)\Delta\alpha\},$$

where

$$\Delta : T(\nu(M)) \to T(\nu(M)) \wedge M_+$$

is the diagonal map, and $T(c)_*$ comes from the Wu$\langle n\rangle$-orientation c of $\nu(M)$. Ψ_h is a quadratic form in the sense that

$$\Psi_h(x+y) - \Psi_h(x) - \Psi_h(y) = 2\cdot(x, y),$$

here $2 : \mathbb{Z}/2 \to \mathbb{Z}/4$ is multiplication by 2. Henceforth Ψ_h *will be referred to as Brown's quadratic form.*

Brown associates an invariant $\sigma(\Psi_h) \in \mathbb{Z}/8$ to Ψ_h, such that if Ψ_h has the following factorization

$$H^n(M; \mathbb{Z}/2) \xrightarrow{q} \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4,$$

where q is a quadratic form in the sense that $q(x+y) - q(x) - q(y) = (x, y)$ is a bilinear form, then

$$\sigma(\Psi_h) = 4(\mathrm{Arf}(q)). \tag{1.5}$$

Notation 1.2. We will denote $\sigma(\Psi_h)$ by $\sigma(x, c, h)$.

We now recall briefly the result on changing orientations [5, p. 299]. Let c_1 and c_2 be two Wu$\langle n\rangle$-orientations of the normal bundle $\nu(M)$ of M^{2n}. Let they differ by a map

$$x : M \to \prod_{i \geqslant n} K(\mathbb{Z}/2, i),$$

where x represents a sequence $\{x_i\}$, $x_i \in H^i(M; \mathbb{Z}/2)$, $i \geqslant n$, then we have the following result of Brown:

$$\Psi_{h_{c_1}}(u) - \Psi_{h_{c_2}}(u) = 2\cdot(u \cup x_n)[M] + \sum_{i>p} (u \cup x_i)[M]\cdot\alpha_{i-p}, \tag{1.6}$$

where $u \in H^p(M; \mathbb{Z}/2)$. Here we are considering $\Psi_{h_{c_\lambda}}$ to take values in a polynomial ring over $\mathbb{Z}/4$ on indeterminates t and α_k, $k = 0, 1, 2, \ldots$, modulo the relations $t\alpha_k = \alpha_{k-1}$, $\alpha_0 = 2$ and $2t = 2\alpha_k = \alpha_k\alpha_1 = 0$ (see [5]).

Definition 1.3. A Wu$\langle n\rangle$-orientation c of $\nu(M)$ is *preferred* if $\Psi_{h_c}(H^p(M; \mathbb{Z}/2)) = 0$ for $p < n$.

Properties 1.4 (Brown).

(i) *Any Wu$\langle n\rangle$-orientation of $\nu(M)$ can be changed to a preferred orientation.*
(ii) *The product of preferred orientations is preferred.*
(iii) *Two preferred orientations over the same classifying map for $\nu(M)$ differ by a trivialization of v_{n+1}.*

2. Statements of the main results

With the notations developed in the last section we are now in a position to state the main results of the paper. The following result tells about the nature of Brown's quadratic form when restricted to the domain of Browder's quadratic form.

Proposition 2.1. *$\Psi_h \mid [\mathrm{Ker}(g^* \circ \Sigma^k)]^n : [\mathrm{Ker}(g^* \circ \Sigma^k)]^n \to \mathbb{Z}/4$ is independent of h and it factors through $\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4$.*

The next theorem relates Browder's quadratic form with Brown's quadratic form when restricted to the domain of Browder's quadratic form.

Theorem 2.2. *If $x \in [\mathrm{Ker}(g^* \circ \Sigma^k)]^n$, then $\Psi_h(x) = 2\Psi(x)$.*

As a consequence of Theorem 2.2 we get a relationship between the generalized Kervaire's invariants of Brown and Browder.

Corollary 2.3. *$\sigma(M, c, h) = 4(k(m, c)) + \sigma(\Psi_h \mid Q)$, for some suitable subspace Q of $H^n(M; \mathbb{Z}/2)$.*

The above results extend the results known for framed manifolds.

Theorem 2.4. *Given a Wu-oriented manifold (M, c) we can always change the orientation c suitably such that $k(M, c)$ is defined.*

Let (M_i, c_i), $i = 1, 2$, be Wu-oriented manifolds and let $c_1 \times c_2$ be an orientation of $M_1 \times M_2$, with the help of which one can define a Browder-type quadratic form and can talk of the generalized Kervaire invariant (this procedure being similar to the one in [3] where a product orientation was defined and used to define the generalized Kervaire invariant). Then:

Theorem 2.5. *$k(M_1 \times M_2, c_1 \times c_2)$ is defined if and only if all characteristic numbers of M_λ involving $v_{n_{\lambda+1}} - n_\lambda + \mu$ vanish whenever $\Psi_\lambda^\mu(x) \neq 0$ for some*

$$x \in \left[\mathrm{Ker}\left(g_\lambda^* \circ \Sigma^k\right)\right]^\mu \cap c_{W_\lambda}^* H^\mu(W_\lambda; \mathbb{Z}/2), \quad \lambda = 1, 2$$

($\lambda + 1$ calculated module 2), $\mu < n_\lambda$.

The notations in the above theorem are self-explanatory (see also Section 5 for these and further notations used in the next theorem, which gives a product formula for the Browder's generalizations of the Kervaire invariant).

Theorem 2.6. *If* $k(M_1 \times M_2,\ c_1 \times c_2)$ *and* $k(M_i,\ c_i)$, $i = 1, 2$, *are defined, then*

$$k(M_1 \times M_2,\ c_1 \times c_2) = k(M_1,\ c_1)\chi(M_2) + k(M_2,\ c_2)\chi(M_1) + \sum_{i=0}^{\min(n_1,n_2)-1} k_i,$$

where $k_i = \mathrm{Arf}(\Psi \mid R_1^i \otimes P_2^{n_1+n_2-i} + P_1^{2n_1-i} \otimes R_2^{n_2-n_1+1})$, R_λ^μ, $\lambda = 1, 2$, *and* P_λ^μ, $\lambda = 1, 2$, *are some suitable subspaces of* $H^\mu(M_\lambda;\ \mathbb{Z}/2)$.

Note. By Theorem 2.4 we can always assume that $k(M_i,\ c_i)$, $i = 1, 2$ are defined.

The following corollaries, whose proofs are sketched in Section 5, justify the study of $k(M_1 \times M_2,\ c_1 \times c_2)$.

Corollary 2.7 (Browder [3]). *If* $(M_i,\ c_i)$, $i = 1, 2$, *are Wu-oriented manifolds with* M_1 *a* π*-manifold, then* $k(M_1 \times M_2,\ e) = k(M_1 \times M_2,\ c_1 \times c_2)$ *is defined and*

$$k(M_1 \times M_2,\ e) = k(M_1,\ c_1)\chi(M_2).$$

Corollary 2.8. *If* $(M_i,\ c_i)$, $i = 1, 2$, *are Wu-oriented manifolds such that the normal bundle of* M_i *restricted to the* ((dim M_i)/2)*-skeleton is trivial*, $i = 1, 2$, *and*

(i) dim $M_1 \neq$ dim M_2 *or*

(ii) dim $M_1 =$ dim M_2 *but* $c_{1_*}[M_1] \neq c_{2_*}[M_2]$. *Then* $k(M_1 \times M_2,\ c_1 \times c_2)$ *is defined and*

$$k(M_1 \times M_2,\ c_1 \times c_2) = k(M_i,\ c_i)\chi(M_{i+1}).$$

(($i + 1$) *calculated modulo 2*).

Corollary 2.9. *If* $(M_i,\ c_i)$, $i = 1, 2$, *are Wu-oriented manifolds one of which has a weak complex structure; if the parity of* (dim M_1)/2 *and* (dim M_2)/2 *are different; and if the normal bundle of* M_i *is trivial when restricted to the* [{(dim M_i)/2} − 1]*-skeleton*, $i = 1, 2$. *Then* $k(M_1 \times M_2,\ c_1 \times c_2)$ *is defined and*

$$k(M_1 \times M_2,\ c_1 \times c_2) = k(M_1,\ c_1)\chi(M_2) + k(M_2,\ c_2)\chi(M_1).$$

We now come to those results which rely on the choices of suitable Wu-orientations on the manifolds in question.

Theorem 2.10. *If* $(M_i,\ c_i)$, $i = 1, 2$, *are Wu-oriented manifolds with preferred orientations, and if* $k(M_i,\ c_i)$, $i = 1, 2$, *are defined, then*

$$k(M_1 \times M_2,\ c_1 \times c_2) = k(M_1,\ c_1)\chi(M_2) + k(M_2,\ c_2)\chi(M_1).$$

Theorem 2.11. $k(\mathbb{C}P^m \times \mathbb{R}P^{2n},\ e) = 0$ *for any product* $Wu\langle v_{m+n+1}\rangle$*-orientation* e.

Theorem 2.12. $k(\mathbb{R}P^{2n} \times \mathbb{R}P^{2n},\ e)$ *is defined if one of the factors has a preferred orientation but not both are having preferred orientations.* (e *being the product orientation.*)

Theorem 2.13. *If $k(\mathbb{R}P^{2n} \times \mathbb{R}P^{2n}, e)$ is defined for some product orientation e, then it is equal to zero.*

Theorem 2.14. *$k(\mathbb{R}P^{2n_1} \times \mathbb{R}P^{2n_2}, e)$ is defined and is equal to zero if*
(i) *$n_2 \geqslant 3n_1$ and e is any product $Wu\langle v_{n_1+n_2+1}\rangle$-orientation.*
(ii) *$n_2 < 3n_1$ and e is a product of preferred orientations.*

3. Proofs of Theorems 2.4, 2.10 and 2.11

Proof of Theorem 2.4. Let (M^{2n}, c) be a Wu$\langle n\rangle$-oriented manifold. If $k(M, c)$ is defined, then there is nothing to show. If $k(M, c)$ is not defined, then $\Psi \mid R \neq 0$, where Ψ is Browder's quadratic form as in (1.1) and R is the radical of the associated bilinear form as given in Condition 1.1 and (1.3). By the definition of R, $\Psi \mid R : R \to \mathbb{Z}/2$ is a nonzero linear map. Extend it to a linear map $L : H^n(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ defined by $L(u) = ((\Psi \mid R) \circ \pi)(u)$, where π is the projection of $H^n(M; \mathbb{Z}/2)$ onto the summand R. By Poincaré duality we get an element $x \in H^n(M; \mathbb{Z}/2)$ such that

$$L(u) = (u \cup x)[M] \quad \forall u \in H^n(M; \mathbb{Z}/2).$$

Change the orientation c by $\{x\}$ to get a new orientation c'. Then by (1.6)

$$\Psi_{h_{c'}}(u) = \Psi_{h_c}(u) + 2 \cdot (u \cup x)[M] \quad \forall u \in H^n(M; \mathbb{Z}/2).$$

Now note that

$$\left[\operatorname{Ker}(g^* \circ \Sigma^k)\right]^n = \left[\operatorname{Ker}(g'^* \circ \Sigma^k)\right]^n = K \text{ (say)},$$
$$\operatorname{Im} c^* = \operatorname{Im} c'^*.$$

This is obtained by using the classifying map of $\nu(M)$ and the Serre exact sequence. Therefore, if $u \in K$ then, by Theorem 2.2,

$$2\Psi'(u) = \Psi_{h_{c'}}(u) = \Psi_{h_c}(u) + 2 \cdot (u \cup x)[M] = 2\Psi(u) + 2(u \cup x)[M],$$

and, if $u \in R = \operatorname{Im} c^* \cap K = \operatorname{Im} c'^* \cap K$, then $2\Psi'(u) = 0$, or $\Psi'(u) = 0$ (as $\Psi(u) = (u \cup x)[M]$ by the definition of L). Therefore, by Condition 1.1, $k(M, c')$ is defined, and Theorem 2.4 is proved. □

Proof of Theorem 2.10. If (M_i, c_i), $i = 1, 2$, are Wu-oriented manifolds with preferred orientations, and if $k(M_i, c_i)$, $i = 1, 2$, are defined, then, by Theorem 2.4 and Definition 1.3, $k(M_1 \times M_2, c_1 \times c_2)$ is defined and, by Theorem 2.6, we get

$$k(M_1 \times M_2, c_1 \times c_2)$$
$$= k(M_1, c_1)\chi(M_2) + k(M_2, c_2)\chi(M_1) + \sum_{i=0}^{\min(n_1,n_2)-1} k_i,$$

where $k_i = \operatorname{Arf}(\Psi \mid R_1^i \otimes P_2^{n_1+n_2-i} + P_1^{2n_1-i} \otimes R_2^{n_2-n_1+1})$,

$$R_\lambda^\mu = \left[\operatorname{Ker}(g_\lambda \circ \Sigma^k)\right]^\mu \cap c_{W_\lambda}^* H^\mu(W_\lambda; \mathbb{Z}/2), \quad \lambda = 1, 2,$$

and P_λ^* is given by

$$H^*(M_\lambda; \mathbb{Z}/2) = c_{W_\lambda}^* H^*(W_\lambda; \mathbb{Z}/2) \oplus S_\lambda^* \oplus P_\lambda^*,$$

where S_λ^* is a direct complement of R_λ^* in $[\operatorname{Ker}(g_\lambda \circ \Sigma^k)]^*$, $\lambda = 1, 2$. Now, if

$$x \otimes y \in R^i \otimes P^{n_1+n_2-i}, \quad i \leqslant \min(n_1, n_2) - 1,$$
$$\Psi(x \otimes y) = \Psi_1^i(x) \otimes \left[\mathrm{Sq}^{n_2-n_1+i}(y)\right][M],$$

where

$$\Psi_1^i(x) = \left[\mathrm{Sq}^{2n_1+1-i}_{((\Sigma\phi)\circ g_1)}(\Sigma^k\iota)\right][W_1]$$

(compare (1.1), see also Section 5). As c_1 is preferred, $\Psi_1^i(x) = \Psi_{h_{c_1}}(x) = 0$. Therefore $\Psi(x \otimes y) = 0$. Thus Ψ takes every generator of $R_1^i \otimes P^{n_1+n_2-i}$ to zero. Similarly Ψ takes every generator of $P^{2n_1-i} \otimes R^{n_2-n_1+i}$ to zero. Therefore $k_i = 0$ for every $i \leqslant \min(n_1, n_2) - 1$, and the proof of Theorem 2.10 is complete. □

Proof of Theorem 2.11. Let $e : \nu(\mathbb{C}P^m \times \mathbb{R}P^{2n}) \to \gamma\langle v_{m+n+1}\rangle$ be the $\mathrm{Wu}\langle v_{m+n+1}\rangle$-orientation given by the composite

$$\nu(\mathbb{C}P^m \times \mathbb{R}P^{2n}) \xrightarrow{\bar{c}_1 \times \bar{c}_2} \gamma\langle m\rangle \times \gamma\langle n\rangle \xrightarrow{\bar{\mu}} \gamma\langle m+n\rangle \xrightarrow{\rho} \gamma\langle v_{m+n+1}\rangle,$$

as given in [3]. For brevity let us use the same notations for the maps on the base spaces:

$$\mathbb{C}P^m \times \mathbb{R}P^{2n} \xrightarrow{\bar{c}_1 \times \bar{c}_2} \mathrm{BO}\langle m\rangle \times \mathrm{BO}\langle n\rangle \xrightarrow{\bar{\mu}} \mathrm{BO}\langle m+n\rangle \xrightarrow{\rho} \mathrm{BO}\langle v_{m+n+1}\rangle.$$

Let

$$g : W/\partial W \to \Sigma^k(\mathbb{C}P^m \times \mathbb{R}P^{2n})_+$$

be the spectral orientation corresponding to e. We prove the theorem by showing that

$$\left[\operatorname{Ker}(g^* \circ \Sigma^k)\right]^{m+n} = 0.$$

By Spanier–Whitehead duality it is enough to prove that

$$\left[\operatorname{Ker}(\rho_* \circ \bar{\mu}_* \circ (\bar{c}_1 \times \bar{c}_2)_*)\right]_{m+n} = 0.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{BO}\langle m\rangle \times \mathrm{BO}\langle n\rangle \arrow[r, "\bar{\mu}"] \arrow[d, "\rho_1 \times \rho_2"] & \mathrm{BO}\langle m+n\rangle \arrow[d, "\rho"] \\
 & \mathrm{BO}\langle v_{m+1}\rangle \times \mathrm{BO}\langle v_{n+1}\rangle \arrow[d, "\pi \times \pi"] & \mathrm{BO}\langle v_{m+n+1}\rangle \arrow[d, "\pi"] \\
\mathbb{C}P^m \times \mathbb{R}P^{2n} \arrow[uur, "\bar{c}_1 \times \bar{c}_2"] \arrow[ur, "c_1 \times c_2"] \arrow[r, "\alpha_1 \times \alpha_2"] & \mathrm{BO} \times \mathrm{BO} \arrow[r, "\mu"] & \mathrm{BO}
\end{tikzcd}$$

where α_1 and α_2 classify $\nu(\mathbb{C}P^m)$ and $\nu(\mathbb{R}P^{2n})$ respectively. By the commutativity of the diagram it follows that

$$\begin{aligned}\left[\operatorname{Ker}(\rho_* \circ \bar{\mu}_* \circ (\bar{c}_1 \times \bar{c}_2)_*)\right]_{m+n} &\subseteq \left[\operatorname{Ker}(\pi_* \circ \rho_* \circ \bar{\mu}_* \circ (\bar{c}_1 \times \bar{c}_2)_*)\right]_{m+n} \\ &= \left[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{m+n}.\end{aligned}$$

So, it is enough to show that

$$\left[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{m+n} = 0.$$

Now,

$$\begin{aligned}(\alpha_1 \times \alpha_2)_* H_{m+n}(\mathbb{C}P^m \times \mathbb{R}P^{2n}; \mathbb{Z}/2) &= \sum_{i=0}^{m} (\mathbb{Z}/2)e_i^2 \otimes (\mathbb{Z}/2)e_{m+n-2i} \\ &\subseteq H_{m+n}(\mathrm{BO} \times \mathrm{BO}; \mathbb{Z}/2),\end{aligned}$$

where $H_*(\mathrm{BO}; \mathbb{Z}/2)$ is the polynomial algebra $\mathbb{Z}/2[e_1, e_2, \ldots]$, $e_i \in H_i(\mathrm{BO}; \mathbb{Z}/2)$. ${\alpha_1}_*$ sends the generator of $H_{2i}(\mathbb{C}P^m; \mathbb{Z}/2)$ to e_i^2; ${\alpha_2}_*$ sends the generator of $H_i(\mathbb{R}P^{2n}; \mathbb{Z}/2)$ to e_i (see [7]).

Since $e_1, e_2, \ldots$ are algebraically independent we get

$$(\alpha_1 \times \alpha_2)_* H_{m+n}(\mathbb{C}P^m \times \mathbb{R}P^{2n}; \mathbb{Z}/2) \cap [\operatorname{Ker} \bar{\mu}_*]_{m+n} = 0.$$

Therefore, as $(\alpha_1 \times \alpha_2)_*$ is injective,

$$\begin{aligned}\left[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{m+n} &= (\alpha_1 \times \alpha_2)_*^{-1}[\operatorname{Ker} \mu_*]_{m+n} \\ &= \left[\operatorname{Ker}(\alpha_1 \times \alpha_2)_*\right]_{m+n} = \{0\}.\end{aligned}$$

This proves Theorem 2.11. □

Proofs of Theorems 2.12 and 2.13. Using the same notations as in the proof of Theorem 2.11, except that $\mathbb{C}P^m$ is to be replaced by $\mathbb{R}P^{2n}$, we get

$$\begin{aligned}\left[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{2n} = (\mathbb{Z}/2)\{&1 \otimes e_{2n} + e_{2n} \otimes 1,\ e_1 \otimes e_{2n-1} \\ &+ e_{2n-1} \otimes e_1, \ldots, e_{n-1} \otimes e_{n+1} + e_{n+1} \otimes e_{n-1}\}.\end{aligned}$$

It can be checked that $[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)]_{2n}$ is self-annihilating under the intersection pairing of $H_{2n}(\mathbb{R}P^{2n} \times \mathbb{R}P^{2n}; \mathbb{Z}/2)$. Therefore,

$$\left[\operatorname{Ker}(\rho_* \circ \bar{\mu}_* \circ (\bar{c}_1 \times \bar{c}_2)_*)\right]_{2n} \subseteq \left[\operatorname{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{2n},$$

is self-annihilating, and so, by Spanier–Whitehead duality, $[\operatorname{Ker}(g^* \circ \Sigma^k)]^{2n}$ is self-annihilating under the cup product pairing. Theorem 2.13 therefore follows from Condition 1.1.

To prove Theorem 2.12 we first note, using Poincaré duality, that

$$\begin{aligned}\left[\operatorname{Ker}(g^* \circ \Sigma^k)\right]^{2n} \subseteq (\mathbb{Z}/2)\{&a^{2n} \otimes 1 + 1 \otimes a^{2n}, \ldots, a^{2n-i} \otimes a^i + a^i \otimes a^{2n-i}, \\ &\ldots, a^{n+1} \otimes a^{n-1} + a^{n-1} \otimes a^{n+1}\},\end{aligned}$$

where $a \in H^1(\mathbb{R}P^{2n}; \mathbb{Z}/2)$ is the generator.

Let us take c_1 to be a preferred orientation of $\nu(\mathbb{R}P^{2n})$ and construct an orientation c_2 of $\nu(\mathbb{R}P^{2n})$ in Wu$\langle n\rangle$-theory by changing c_1 by $\{a^{n+1}, a^{n+2}, \dots, a^{2n}\}$. Then

$$\begin{aligned}
&\Psi_h(a^{2n-i}\otimes a^i + a^i\otimes a^{2n-i})\\
&\quad= \Psi_h(a^{2n-i}\otimes a^i) + \Psi_h(a^i\otimes a^{2n-i}) + 2[a^{2n}\otimes a^{2n}][\mathbb{R}P^{2n}\times\mathbb{R}P^{2n}]\\
&\quad= 2+0+2\equiv 0 \pmod 4,
\end{aligned}$$

because c_1 is preferred (see (1.4) and [4, p. 299]). Hence, by Theorem 2.2,

$$\Psi(a^{2n-i}\otimes a^i + a^i\otimes a^{2n-i}) = 0 \quad \forall i < n_1,$$

implying that $\Psi \mid R \equiv 0$, and thus, by Condition 1.1, $k(\mathbb{R}P^{2n}\times\mathbb{R}P^{2n}, e)$ is defined (e is the product orientation obtained by c_1 and c_2).

If we take both c_1 and c_2 to be preferred orientations, then

$$\Psi_h(a^{2n-i}\otimes a^i + a^i\otimes a^{2n-i}) = 0+0+2\equiv 2 \pmod 4.$$

That is, $\Psi \mid R \neq 0$, and therefore $k(\mathbb{R}P^{2n}\times\mathbb{R}P^{2n}, e)$ is not defined. This completes the proof of Theorem 2.12. □

Proof of Theorem 2.14. Let $g: W/\partial W \to \Sigma^{2k}(\mathbb{R}P^{2n_1}\times\mathbb{R}P^{2n_2})_+$ be the spectral orientation map obtained from a product Wu$\langle v_{n_1+n_2+1}\rangle$-orientation on $\nu(\mathbb{R}P^{2n_1}\times\mathbb{R}P^{2n_2})$. By similar calculations as in the previous theorems one obtains:

$$[\operatorname{Ker}(g^*\circ\Sigma^{2k})]^{n_1+n_2}\begin{cases}
=0, & \text{if } n_2\geqslant 3n_1,\\
\subseteq(\mathbb{Z}/2)\{1\otimes a^{n_1+n_2}+a^{3n_1-n_2}\otimes a^{2(n_2-n_1)}, \\
\quad \dots, a^{(3n_1-n_2)/2-1}\otimes a^{(3n_2-n_1)/2+1} \\
\quad +a^{(3n_1-n_2)/2+1}\otimes a^{(3n_2-n_1)/2-1}\}, \\
\quad \text{if } n_2<3n_1 \text{ and } 2\mid(n_1+n_2),\\
\subseteq(\mathbb{Z}/2)\{1\otimes a^{n_1+n_2}+a^{3n_1-n_2}\otimes a^{2(n_2-n_1)}, \\
\quad \dots, a^{(3n_1-n_2-1)/2}\otimes a^{(3n_2-n_1+1)/2} \\
\quad +a^{(3n_1-n_2+1)/2}\otimes a^{(3n_2-n_1-1)/2}\}, \\
\quad \text{if } n_2<3n_1 \text{ and } 2\nmid(n_1+n_2).
\end{cases} \tag{3.1}$$

This finishes the proof of Theorem 2.14(i).

To prove Theorem 2.14(ii) we take $n_1\neq n_2$, $n_2<3n_1$, c_λ a preferred Wu$\langle n\rangle$-orientation of $\nu(\mathbb{R}P^{2n_\lambda})$, $\lambda=1, 2$, and g obtained from c_1 and c_2. Evaluating Brown's quadratic form Ψ_h on the generators of the right-hand side of (3.1) one obtains:

$$\begin{aligned}
&\Psi_h(a^k\otimes a^{n_1+n_2-k}+a^{3n_1-n_2-k}\otimes a^{2(n_2-n_1)+k})\\
&\quad=\Psi_h(a^k\otimes a^{n_1+n_2-k})+\Psi_h(a^{3n_1-n_2-k}\otimes a^{2(n_2-n_1)+k})\\
&\qquad+2[a^{3n_1-n_2}\otimes a^{3n_2-n_1}][\mathbb{R}P^{2n_1}\times\mathbb{R}P^{2n_2}]\\
&\quad=\begin{cases}1+0+0\equiv 1 \pmod 4, & \text{if } k=n_1,\\ 0+0+0\equiv 0 \pmod 4, & \text{if } k\neq n_1.\end{cases}
\end{aligned}$$

This is true both when $2 \mid (n_1 + n_2)$ and when $2 \nmid (n_1 + n_2)$. However, $a^{n_1} \otimes a^{n_2} + a^{2n_1 - n_2} \otimes a^{2n_2 - n_1} \in$ RHS of (3.1) but it is not an element of $[\mathrm{Ker}(g^* \circ \Sigma^{2k})]^{n_1+n_2}$. (In fact Theorem 2.2 cross-checks this fact.) Therefore, we get that Ψ_h annihilates all the generators of $[\mathrm{Ker}(g^* \circ \Sigma^{2k})]^{n_1+n_2}$. Theorem 2.14(ii) now follows. □

4. Proofs of Proposition 2.1, Theorem 2.2 and Corollary 2.3

Let (M^{2n}, c) be a manifold with a Wu$\langle n \rangle$-orientation on $\nu(M)$. Let

$$g : W/\partial W \to \Sigma^k M_+$$

be the spectral orientation map corresponding to c. Let

$$\bar{g} : W/\partial W \to Q(\Sigma^k M_+)$$

be the adjoint of g, where $Q() = \Omega^\infty \Sigma^\infty()$, then we have:

Lemma 4.1. *If $p : W/\partial W \to S^{2n+k}$ is a degree-one map, and if $r : Q(\Sigma^k K(\mathbb{Z}/2, n)) \to K(\mathbb{Z}/2, n+k)$ is the canonical retraction* (see [6]), *then we have the following short exact sequence*:

$$0 \to [S^{2n+k}, Q(\Sigma^k K(\mathbb{Z}/2, n))] \xrightarrow{p^*} [W/\partial W, Q(\Sigma^k K(\mathbb{Z}/2, n))] \xrightarrow{r^*} H^{n+k}(W, \partial W; \mathbb{Z}/2) \to 0. \tag{4.1}$$

Proof. Let $K(\mathbb{Z}/2, 2n+k) \to E_{n+k} \to K(\mathbb{Z}/2, n+k)$ be the fibration with k-invariant $\mathrm{Sq}^{n+1}\iota_{n+k}$. The homotopy-exact sequence of this fibration gives rise to the following commutative diagram (where in we denote $K(\mathbb{Z}/2, i)$ by K_i for convenience):

$$\begin{tikzcd}
 & \mathbb{Z}/2 \cong [S^{2n+k}, Q(\Sigma^k K_n)] \arrow[d, "\cong"'] & \\
{[S^{2n+k}, K_{2n+k}]} \arrow[r, "\cong"] \arrow[d, "\cong"', "p^*"] & {[S^{2n+k}, E_{n+k}]} \arrow[r] \arrow[d, "p^*"] & {[S^{2n+k}, K_{n+k}]} \arrow[d, "p^*"] \\
{[W/\partial W, K_{2n+k}]} \arrow[r] & {[W/\partial W, E_{n+k}]} \arrow[r] & {[W/\partial W, K_{n+k}]} \\
H^{n+k-1}(W/\partial W; \mathbb{Z}/2) \arrow[u, "\mathrm{Sq}^{n+1}"] & {[W/\partial W, Q(\Sigma^k K_n)]} \arrow[u, "\cong"] \arrow[ur, "r^*"'] &
\end{tikzcd} \tag{4.2}$$

Since $\mathrm{Sq}^{n+1} H^{n+k-1}(W/\partial W; \mathbb{Z}/2) = 0$ by the definition of W, Lemma 4.1 follows by chasing the above diagram (compare [4,6]). □

Proof of Proposition 2.1. Consider the following composite:

$$\begin{aligned}\left[\ker(g^{*}\circ\Sigma^{k})\right]^{n} &\xrightarrow{Q\Sigma^{k}} \left[Q(\Sigma^{k}M_{+}),\, Q(\Sigma^{k}K_{n+})\right]\\ &\xrightarrow{\bar{g}^{*}} \left[W/\partial W,\, Q(\Sigma^{k}K_{n+})\right] \xrightarrow{h} \mathbb{Z}/2,\end{aligned} \tag{4.3}$$

where h has been chosen to make $h\circ p^{*}$ injective, p^{*} is as in (4.2) (see [4,6]), and note that $[W/\partial W, Q(\Sigma^{k}K_{n+})]$ is Spanier–Whitehead dual to $\{S^{2n+k}, T(\gamma(W))\wedge K_{n}\})$. If $x\in[\ker(g^{*}\circ\Sigma^{k})]^{n}$ and r^{*} as in (4.2), then $r^{*}\circ\bar{g}^{*}\circ Q\Sigma^{k}(x)=g^{*}\circ\Sigma^{k}(x)=0$. So by Lemma 4.1, $\bar{g}^{*}\circ Q\Sigma^{k}(x)\in \operatorname{Im} p^{*}$. So $\Psi_{h}(x)=h(\bar{g}^{*}\circ Q\Sigma^{k}(x))$ is independent of the choice of h and lies in the image of $2:\mathbb{Z}/2\to\mathbb{Z}/4$. This completes the proof of Proposition 2.1. □

Proof of Theorem 2.2. Let $x\in[\ker(g^{*}\circ\Sigma^{k})]^{n}$. By Proposition 2.1, $\Psi_{h}(x)=2\cdot q(x)$, where $q:[\ker(g^{*}\circ\Sigma^{k})]^{n}\to\mathbb{Z}/2$. We will show that $q=\Psi$, where Ψ is Browder's quadratic form. To do this we consider the homotopy-theoretic description of $\mathrm{Sq}^{n+1}_{((\Sigma^{k}\phi)\circ g)}(\Sigma^{k}\iota)$, where $\phi: M\to K(\mathbb{Z}/2, n)$ represents x. This is given by the following diagram:

$$\begin{tikzcd}
K_{2n+k} \arrow[rr, "i"] & & E_{n+k} \arrow[d, "\pi"] & \\
W/\partial W \arrow[u, "(\Sigma^{k}\iota)''"] \arrow[r, "(\Sigma^{k}\phi)\circ g"] & \Sigma^{k}K_{n} \arrow[ur, "(\Sigma^{k}\iota)'"] \arrow[r, "\Sigma^{k}\iota"] & K_{n+k} \arrow[r, "\mathrm{Sq}^{n+1}"] & K_{2n+k+1}
\end{tikzcd}$$

where $K_{2n+k}\to E_{n+k}\to K_{n+k}$ is the fibration considered in the proof of Lemma 4.1. $(\Sigma^{k}\iota)'$ is a lift of $(\Sigma^{k}\iota)$, which exist by the properties of Sq^{n+1} and K_{n}.

By using properties of ϕ etc., one can show that

$$i\circ(\Sigma^{k}\iota)''\simeq(\Sigma^{k}\iota)'\circ(\Sigma^{k}\phi)\circ g.$$

(See [8] for details.) The homotopy class of $(\Sigma^{k}\iota)''$ is by definition $\mathrm{Sq}^{n+1}_{((\Sigma^{k}\phi)\circ g)}(\Sigma^{k}\iota)$ modulo indeterminacy, which is zero.

Now we start proving the equality of Ψ and q. Let $\Psi(x)=0$, then $\mathrm{Sq}^{n+1}_{((\Sigma^{k}\phi)\circ g)}(\Sigma^{k}\iota)=0$. So $(\Sigma^{k}\iota)''\simeq 0$. Therefore $(\Sigma^{k}\iota)'\circ(\Sigma^{k}\phi)\circ g\simeq 0$, which means that $\bar{g}^{*}\circ Q(\Sigma^{k}x)=0$. So $\Psi_{h}(x)=0$. Hence $q(x)=0$.

Conversely, let $q(x)=0$. Then $h(\bar{g}^{*}\circ Q(\Sigma^{k}x))=0$. Since $\bar{g}^{*}\circ Q(\Sigma^{k}x)\subseteq \operatorname{Im} p^{*}$ (refer to the proof of Proposition 2.1), and $h\circ p^{*}$ is injective (choice of h in Proposition 2.1), $\bar{g}^{*}\circ Q(\Sigma^{k}x)=0$. So $(\Sigma^{k}\iota)'\circ(\Sigma^{k}\phi)\circ g\simeq 0$ or $i\circ(\Sigma^{k}\iota)''\simeq 0$. But from the commutative diagram

$$\begin{tikzcd}
{[S^{2n+k}, K_{2n+k}]} \arrow[r, "p^{*}"] \arrow[d, "i_{*}", "\cong"'] & {[W/\partial W, K_{2n+k}]} \arrow[d, "i_{\bullet}"] \\
{[S^{2n+k}, E_{n+k}]} \arrow[r, "p^{\bullet}"] & {[W/\partial W, E_{n+k}]}
\end{tikzcd}$$

we see that $i_\bullet \circ p^* \simeq p^\bullet \circ i_*$, where i_* and p^* are isomorphisms and $p^\bullet$ is injective (look at (4.2)). Hence $i_\bullet$ is injective. Therefore $i \circ (\Sigma^k \iota)'' \simeq 0$, consequently $(\Sigma^k \iota)'' \simeq 0$. Therefore $\mathrm{Sq}^{n+1}_{((\Sigma^k \phi)\circ g)}(\Sigma^k \iota) = 0$. Thus $\Psi(x) = 0$, and the proof of Theorem 2.2 is complete. □

Proof of Corollary 2.3. Consider the following direct sum decompositions

$$\left[\ker(g^* \circ \Sigma^k)\right]^n = R^n \oplus S^n,$$
$$H^n(M; \mathbb{Z}/2) = S^n \oplus Q^n,$$

where R^n is the radical of the bilinear form associated to Browder's quadratic form, S^n a direct complement of R^n in the space mentioned, and Q^n is the orthogonal complement of S^n in the space mentioned under the cup product pairing. Now, by definition, $k(M, c) = \mathrm{Arf}(\Psi \mid S)$ and, by Theorem 2.2, $2\,\mathrm{Arf}\,(\Psi \mid S) = \Psi_h \mid S$. Therefore it follows from the result of Brown [4, 1.20] that

$$\sigma(\Psi_h \mid S) = 4\,\mathrm{Arf}(\Psi \mid S) = 4k(M, c).$$

As $S^n \oplus Q^n$ is an orthogonal direct sum, we have by [4]

$$\sigma(M, c) = \sigma(\Psi_h \mid S) + \sigma(\Psi_h \mid Q).$$

Therefore

$$\sigma(M, c) = 4k(M, c) + \sigma(\Psi_h \mid Q)$$

proving Corollary 2.3. □

5. Proofs of Theorems 2.5 and 2.6, and Corollaries 2.7–2.9

We will freely use the standard notations of [1–3] and freely refer to the methods of these papers for the sake of brevity.

Proofs of Corollaries 2.7–2.9. Corollary 2.7 is proved in [3]. For (M_i, c_i), $i = 1, 2$, as considered in Corollaries 2.8 and 2.9, using the relation $V = (\mathrm{Sq})^{-1}W$ of the universal total Wu-class and Stiefel–Whitney classes it follows that

$$v_j(M_i) = 0 \quad \forall j < n_i, \qquad n_i = \dim M_i/2,\ i = 1, 2. \tag{5.1}$$

Moreover, the classifying map $M_i \to B$ of $\nu(M_i)$ lifts to $\alpha_i : M_i \to B(n_i, n_i + 1, \ldots, \infty)$, where $B(n_i, n_i + 1, \ldots, \infty)$ is the $(n_i - 1)$-connective covering of B, $i = 1, 2$. So we can consider c_i also to be maps:

$$c_i : M_i \to B(n_i, n_i + 1, \ldots, \infty)\langle n_i \rangle, \quad i = 1, 2.$$

Using similar notations as used in the proof of Theorem 2.11 replacing B by $B(n_i, n_i + 1, \ldots, \infty)$ one can show that

$$\left[\mathrm{Ker}(\rho_* \circ \bar{\mu}_* \circ (c_1 \times c_2)_*)\right]_{n_1+n_2} \subseteq \left[\mathrm{Ker}(\mu_* \circ (\alpha_1 \times \alpha_2)_*)\right]_{n_1+n_2}. \tag{5.2}$$

Now, for M_i, $i=1, 2$, as in Corollary 2.8(i):

$$(\alpha_1\times\alpha_2)_* H_{n_1+n_2}(M_1\times M_2;\ \mathbb{Z}/2)$$
$$=\begin{cases}(\mathbb{Z}/2)\{1\otimes\alpha_{2*}(y_{n_1+n_2})\mid y_{n_1+n_2}\in H_{n_1+n_2}(M_2;\ \mathbb{Z}/2)\}, & \text{if } n_1<n_2,\\ (\mathbb{Z}/2)\{\alpha_{1*}(x_{n_1+n_2})\otimes 1\mid x_{n_1+n_2}\in H_{n_1+n_2}(M_1;\ \mathbb{Z}/2)\}, & \text{if } n_1>n_2.\end{cases}$$

For M_i, $i=1, 2$, as in Corollary 2.8(ii):

$$(\alpha_1\times\alpha_2)_* H_{n_1+n_2}(M_1\times M_2;\ \mathbb{Z}/2)=(\mathbb{Z}/2)\{1\otimes\alpha_{2*}[M_2]+\alpha_{1*}[M_1]\otimes 1\}.$$

In both the cases the given hypothesis can be used to see that

$$[\operatorname{Im}(\alpha_1\times\alpha_2)_*]_{n_1+n_2}\cap[\operatorname{Ker}\mu_*]_{n_1+n_2}=0.$$

Therefore, as in Theorem 2.11,

$$[\operatorname{Ker}(\rho_*\circ\bar{\mu}_*\circ(c_1\times c_2)_*)]_{n_1+n_2}\subseteq[\operatorname{Ker}(\mu_*\circ(c_1\times c_2)_*)]_{n_1+n_2}.$$

We now concentrate on Corollary 2.8(i). As $v_j(M_i)=0\ \forall j\leqslant n_i$ and $v_j(M_{i+1})=0$ $\forall j\leqslant n_{i+1}$ ($i=1, 2$; $i+1$ calculated module 2), using Theorem 2.5 the definability of $k(M_1\times M_2, c_1\times c_2)$ follows. Also in this case

$$[\operatorname{Ker}(g_\lambda^*\circ\Sigma^k)]^\mu=H^\mu(M_\lambda;\ \mathbb{Z}/2)\quad\forall\mu\leqslant n_\lambda,\quad\lambda=1, 2.$$

Hence $R_\lambda^\mu=0\ \forall\mu\leqslant n_\lambda$, so $k_\mu=0\ \forall\mu\leqslant\min(n_1, n_2)-1$ (see Theorem 2.6), and $\chi(M_\lambda)$ is even, by Poincaré duality, $\lambda=1, 2$. So Corollary 2.8(i) follows.

The proof of Corollary 2.8(ii) is analogous to that of Corollary 2.8(i); so is the proof of Corollary 2.9, once we have noted the following: If $M_1^{2n_1}$ is weakly complex and n_1 is odd, then

$$(\alpha_1\times\alpha_2)_* H_{n_1+n_2}(M_1\times M_2;\ \mathbb{Z}/2)$$
$$=\begin{cases}(\mathbb{Z}/2)\{1\otimes\alpha_{2*}(y_{n_1+n_2})\mid y_{n_1+n_2}\in H_{n_1+n_2}(M_2;\ \mathbb{Z}/2)\}, & \text{if } n_1<n_2,\\ 0, & \text{if } n_1>n_2.\end{cases}$$

If $M_1^{2n_1}$ is weakly complex and n_1 is even, then

$$(\alpha_1\times\alpha_2)_* H_{n_1+n_2}(M_1\times M_2;\ \mathbb{Z}/2)$$
$$=\begin{cases}(\mathbb{Z}/2)\{1\otimes\alpha_{2*}(y_{n_1+n_2})\mid y_{n_1+n_2}\in H_{n_1+n_2}(M_2;\ \mathbb{Z}/2)\}, & \text{if } n_1<n_2,\\ (\mathbb{Z}/2)\{\alpha_{1*}(x_{n_1+n_2})\otimes 1\mid x_{n_1+n_2}\in H_{n_1+n_2}(M_1;\ \mathbb{Z}/2)\}, & \text{if } n_1>n_2.\end{cases}\qquad\square$$

Proofs of Theorems 2.5 and 2.6 (*Sketch*). If

$$g_\lambda: W_\lambda/\partial W_\lambda\to\Sigma^k M_{\lambda+},\quad\lambda=1, 2,$$

are the spectral orientation maps corresponding to the Wu-orientations c_λ, $\lambda=1, 2$, of the manifolds M_λ, $\lambda=1, 2$, then the spectral orientation map for the orientation $c_1\times c_2$ is given by

$$g_1\wedge g_2:(W_1/\partial W_1)\wedge(W_1/\partial W_1)\to\Sigma^{2k}(M_1\times M_2)_+,$$

and Browder's quadratic form,

$$\Psi : \left[\operatorname{Ker}\left((g_1 \wedge g_2)^* \circ \Sigma^{2k}\right)\right]^{n_1+n_2} \to \mathbb{Z}/2, \tag{5.3}$$

is defined by

$$\Psi(x) = \left[\mathrm{Sq}^{n_1+n_2+1}_{((\Sigma^{2k}\Phi)\circ(g_1 \wedge g_2))}(\Sigma^{2k}\iota)\right][W_1 \times W_2].$$

(Notations are as explained in Section 1.) Let us introduce the following notations

$$R^i = \left[\operatorname{Ker}\left((g_1 \wedge g_2)^* \circ \Sigma^{2k}\right)\right]^i \cap (c_{W_1} \times c_{W_2})^* H^i(W_1 \times W_2; \mathbb{Z}/2),$$

$$R^i_\lambda = \left[\operatorname{Ker}(g^*_\lambda \circ \Sigma^k)\right]^i \cap c^*_{W_\lambda} H^i(W_\lambda; \mathbb{Z}/2), \quad \lambda = 1, 2.$$

The proof of Theorem 2.5 follows from a close examination of the restriction of Ψ on the radical of its associated bilinear form $R^{n_1+n_2}$, an appeal to Condition 1.1 and an adaptation of the techniques used in [1–3].

The proof of Theorem 2.6 also depends on an adaptation of the techniques of [1]. The major difference is the following: for the Wu-orientations coming from degree-one maps one gets a nice orthogonal splitting of the domain of Ψ, whereas in the general case under consideration one does not get such a nice splitting. Suppose

$$\left[\operatorname{Ker}\left((g_1 \wedge g_2)^* \circ \Sigma^{2k}\right)\right]^{n_1+n_2} = R^{n_1+n_2} \oplus S^{n_1+n_2}, \tag{5.4}$$

$$\left[\operatorname{Ker}(g^*_\lambda \circ \Sigma^k)\right]^i = R^i_\lambda \oplus S^i_\lambda, \qquad i < 2n_\lambda, \quad \lambda = 1, 2, \tag{5.5}$$

$$H^i(M_\lambda; \mathbb{Z}/2) = c^*_{W_\lambda} H^i(W_\lambda; \mathbb{Z}/2) \oplus T^i_\lambda, \quad i < 2n_\lambda, \quad \lambda = 1, 2, \tag{5.6}$$

$$H^i(M_\lambda; \mathbb{Z}/2) = S^i_\lambda \oplus Q^i_\lambda, \qquad i < 2n_\lambda, \quad \lambda = 1, 2, \tag{5.7}$$

$$T^i_\lambda = S^i_\lambda \oplus P^i_\lambda, \qquad i < 2n_\lambda, \quad \lambda = 1, 2. \tag{5.8}$$

S^*, and S^*_λ can be chosen respectively to be any direct complement of R^*, and R^*_λ in (5.4) and (5.5). It follows that the cup product pairing of $H^*(M_1 \times M_2; \mathbb{Z}/2)$, and $H^*(M; \mathbb{Z}/2)$ restricted to S^* and S^*_λ respectively are nonsingular. T^*_λ in (5.6) can be chosen to contain S^*_λ, for

$$c^*_{W_\lambda} H^i(W_\lambda; \mathbb{Z}/2) \cap S^*_\lambda = \{0\}.$$

As cup product pairing restricted to S^*_λ is nonsingular we can take the splittings (5.7) and (5.8) to be orthogonal. Finally we note that R^*_λ is a subspace of Q^*_λ.

We first note that

$$\operatorname{Arf}(\Psi \mid S^{n_1+n_2}) = \operatorname{Arf}(\overline{\Psi}) = k(M_1 \times M_2, \bar{c}_1 \times \bar{c}_2).$$

Now using the above splitting we can write

$$S^{n_1+n_2} \cong S^*_1 \otimes S^*_2 \oplus Q^*_1 \otimes S^*_2 \oplus S^*_1 \otimes Q^*_2 \oplus (R^*_1 \otimes P^*_2 \oplus P^*_1 \otimes R^*_2). \tag{5.9}$$

As both $S^{n_1+n_2}$ and the right-hand side of (5.9) are direct complements of R and as Ψ does not depend on the choice of direct complements of R we can treat "$\cong$" as an equality. Also since all the summands of (5.9) are mutually orthogonal

$$\begin{aligned}\operatorname{Arf}(\Psi \mid S) = {} & \operatorname{Arf}(\Psi \mid S_1^* \otimes S_2^*) + \operatorname{Arf}(\Psi \mid Q_1^* \otimes S_2^*) \\ & + \operatorname{Arf}(\Psi \mid S_1^* \otimes Q_2^*) + \operatorname{Arf}(\Psi \mid (R_1^* \otimes P_2^* \oplus P_1^* \otimes R_2^*)).\end{aligned}$$

The rest of the proof of Theorem 2.5 is more or less an adaptation of [1]. □

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